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ON THE EXPANSION OF AN ARBITRARY FUNCTION IN A SERIES OF EIGENFUNCTIONS OF BOUNDARY VALUE PROBLEMS II*

BY

S. MINAKSHISUNDARAM, *Madras University*

[Received 5 February, 1941].

Let D be a bounded open domain, simply or multiply connected, whose boundary C is composed of a finite number of regular curves in the x, y plane, and let $\omega_n(x, y)$ be the eigenfunctions (normal and orthogonal) with the corresponding eigenvalues $\lambda_n > 0$, $n = 1, 2, \ldots$ of the boundary value problem

$$\Delta\omega + \lambda\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} + \lambda\omega = 0 \tag{1}$$

$$\omega(x, y) = 0 \text{ on } C. \tag{2}$$

An important problem is to determine the possibility of expanding an arbitrary function $f(x, y)$ defined in D in a series of eigenfunctions of (1). In other words if $f(x, y)$ is integrable in D and if

$$f(x, y) \sim \sum a_n \omega_n(x, y) \tag{3}$$

$$a_n = \iint_D f(x, y)\,\omega_n(x, y)\,dx\,dy, \tag{4}$$

it is required to study the convergence or the summability of the series on the right of (3).

It is well known† that, *if $f(x, y)$ is continuous in D satisfying the boundary condition* (2) *and possesses continuous*

* I Paper, *Proc. Ind. Acad. Sc.* 12 (1940), 462-5.

† Courant and Hilbert, *Mathematische Physik* I, 2 aufl. p. 319.

derivatives of the first two orders, then the series on the right of (3) converges absolutely and uniformly to the function on the left.

In my first paper, I have shown that

If $f(x, y)$ is any bounded measurable function defined in D, then at any interior point (x, y) of D, where $f(x, y)$ is continuous

$$f(x, y) = \lim_{\varepsilon=0} \sum a_n \omega_n(x, y) \exp(-\lambda_n t), \tag{5}$$

the series on the right being absolutely convergent for $t > 0$.

The main purpose of this note is to prove the following theorem on the convergence of the Fourier expansion.

THEOREM 1. *If $f(x, y) \varepsilon L$ in D, and also if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and εL^2 in D so that*

$$\sum \lambda_n a_n^2 < \infty, \tag{6}$$

then at all points (x, y) of D where $f(x, y)$ is continuous

$$f(x, y) = \sum a_n \omega_n(x, y), \tag{7}$$

the series on the right being convergent.

It is clear from this theorem that if $f(x, y)$ fulfills the conditions of Theorem 1, the Fourier series of $f(x, y)$ is convergent almost everywhere in D.

It is quite likely that Theorem 1 has its analogue in spaces of higher dimensions as well. But the method of this paper, while it can be adopted to the corresponding problem in three dimensions, cannot be extended to higher dimensions without further modifications. The following seems to be an apparent generalization of Theorem 1.

Let $\omega_n(x_1, x_2, .., x_m)$ be the eigenfunctions of

$$\Delta\omega + \lambda\omega \equiv \sum_{i=1}^{m} \frac{\partial^2 \omega}{\partial x_i^2} + \lambda\omega = 0 \tag{8}$$

$$\omega(x_1, \ldots, x_m) = 0 \text{ on } \Gamma, \tag{9}$$

where Γ is the boundary of a bounded m-dimensional domain Ω, and let $\lambda_n > 0$, $n = 1, 2, \ldots$ be the eigenvalues corresponding to the eigenfunctions $\omega_n(x_1, \ldots, x_m)$.

Let $f(x_1, x_2, \ldots, x_m)$ be an arbitrary function integrable in Ω and let

$$f(x_1, \ldots, x_m) \sim \sum a_n \omega_n(x_1, \ldots, x_m) \tag{10}$$

$$a_n = \int_\Omega f \omega_n dx_1 \ldots dx_m. \tag{11}$$

Then we have

THEOREM 2. *If*

$$\sum \lambda_n^{m/2} a_n^2 < \infty, \tag{12}$$

then at any interior point of Ω, *where* $f(x_1, x_2, \ldots, x_m)$ *is continuous, the series*

$$\sum a_n \omega_n(x_1, \ldots, x_m)$$

converges to the value of the function at the point.

Here we prove this theorem for $m = 2$ and 3 only and leave the question open for $m > 3$. The paper then concludes with a few remarks on the obvious extension of this theorem to other boundary value problems etc.

PROOF OF THEOREM 1. Let

$$\begin{aligned} \psi_R(x, y) &= \frac{1}{2\pi} \log \frac{r}{R} + \frac{1}{4\pi}\left(1 - \frac{r^2}{R^2}\right), \ r \leqslant R \\ &= 0, \ r \geqslant R, \end{aligned} \tag{13}$$

where $r^2 = (x-x_0)^2 + (y-y_0)^2$. Then, if K_R is a circle with centre (x_0, y_0) and radius R, we have*,

$$H[\psi_R] \leqslant \tau R^2, \tag{14}$$

where

$$H[\phi] = \iint_D \phi^2 \, dxdy \tag{15}$$

* Courant and Hilbert, *Mathematische Physik II*, p. 500.

and τ is a constant. If $\phi(x, y)$ is any function defined in D and differentiable twice, then

$$\phi(x_0, y_0) = \frac{1}{\pi R^2}\iint_{K_R} \phi(x, y)\, dxdy + \iint_{K_R} \psi_R \Delta\phi dxdy. \quad (16)$$

Therefore

$$\left| \phi(x_0, y_0) - \frac{1}{\pi R^2}\iint_{K_R} \phi(x, y)\, dx\, dy \right|^2 \leqslant \tau R^2 H_{K_R}[\Delta\phi]. \quad (17)$$

Now, if $f(x, y)$ is an arbitrary function defined and integrable in D, and satisfying (3) and (4), and if we set

$$S_n(x, y) = \sum_{\nu=1}^{n} a_\nu \omega_\nu(x, y) \quad (18)$$

and

$$M_R[f] = \frac{1}{\tau R^2}\iint_{K_R} f(x, y)\, dxdy, \quad (19)$$

then

$$f(x_0, y_0) - S_n(x_0, y_0) = f - M_R[f] + M_R[f - S_n] + M_R[S_n] - S_n \quad (20)$$

so that

$$|f - S_n| \leqslant |f - M_R[f]| + |M_R[f - S_n]| + |S_n - M_R[S_n]|. \quad (21)$$

By (17), $|S_n - M_R[S_n]|^2 \leqslant \tau R^2 H_{K_R}[\Delta S_n] \leqslant \tau R^2 H[\Delta S_n]$

$$\leqslant \tau R^2 \sum_{\nu=1}^{n} \lambda_\nu^2 a_\nu^2 \quad (22)$$

so that

$$|S_n - M_R[S_n]| \leqslant \tau^{\frac{1}{2}} R \left(\sum_{\nu=1}^{n} \lambda_\nu^2 a_\nu^2 \right)^{\frac{1}{2}} \quad (23)$$

Again

$$|M_R[f - S_n]|^2 = \left| \frac{1}{\pi R^2}\iint_{K_R} (f - S_n)\, dxdy \right|^2$$

$$\leqslant \frac{1}{\pi R^2}\iint_{K_R} |f - S_n|^2\, dxdy \leqslant \frac{1}{\pi R^2} \sum_{\nu=n+1}^{\infty} a_\nu^2, \quad (24)$$

so that

$$|M_R[f - S_n]| \leqslant \frac{1}{\pi^{\frac{1}{2}} R}\left(\sum_{\nu=n+1}^{\infty} a_\nu^2 \right)^{\frac{1}{2}}. \quad (25)$$

Now, using (6), we have

$$\frac{1}{\lambda_n}\sum_{\nu=1}^{n}\lambda_\nu^2 a_\nu^2 = o\,(1). \tag{26}$$

Also, if we set

$$\gamma_n = \sum_{\nu=1}^{n}\lambda_\nu a_\nu^2 \tag{27}$$

$$\gamma = \sum_{\nu=1}^{\infty}\lambda_\nu a_\nu^2 \tag{28}$$

we observe that

$$\sum_{n+1}^{\infty} a_\nu^2 = \sum_{n+1}^{\infty}\frac{\gamma_\nu - \gamma_{\nu-1}}{\lambda_\nu} = \sum_{n+1}^{\infty}\frac{(\gamma_\nu-\gamma)-(\gamma_{\nu-1}-\gamma)}{\lambda_\nu}$$

$$= -\frac{(\gamma_n-\gamma)}{\lambda_n} + \sum_{n+1}^{\infty}(\gamma_\nu-\gamma)\left(\frac{1}{\lambda_\nu}-\frac{1}{\lambda_{\nu+1}}\right)$$

$$= o\left(\frac{1}{\lambda_n}\right). \tag{29}$$

Hence, from (21), (23), (25), (26) and (29), we observe that

$$|f-S_n| \leqslant |f-M_R[f]| + o\left(\frac{1}{R\lambda_n^{\frac{1}{2}}}\right) + o(R\lambda_n^{\frac{1}{2}}). \tag{30}$$

Choosing $R = 1/\lambda_n^{\frac{1}{2}}$ and letting $n \to \infty$, we obtain (7) at a point of continuity of $f(x, y)$. From the above argument it is clear that even at a point where

$$\lim_{R=0} M_R[f] \tag{31}$$

exists, the Fourier series of $f(x, y)$ converges to this value.

PROOF OF THEOREM 2 FOR $m=3$.

In the above proof the inequality (14) played an important role. But if the number m of the independent variables $x_1, \ldots, x_m$ is greater than 2, we have to take

$$\psi_R(x_1, x_2, \ldots, x_m) = -\frac{1}{(m-2)\omega_m}\left[\frac{1}{r^{m-2}} - \frac{m}{2}\frac{1}{R^{m-2}} + \frac{m-2}{2}\frac{r^2}{R^m}\right] \tag{32}$$

in place of the function in (13) and then the inequality (14) will not in general hold good. However if $m=3$ we have the following inequality,

$$H[\psi_R] \leqslant \tau R. \tag{33}$$

The relation (33) ceases to hold good if $m > 3$. Using this inequality, we have instead of (23) and (25) the following

$$|S_n - M_R[S_n]| \leqslant \left(\tau R \sum_{\nu=1}^{n} \lambda_\nu^2 a_\nu \right)^{\frac{1}{2}} \tag{34}$$

$$|M_R[f - S_n]| \leqslant \left(\frac{3}{4\tau R^3} \sum_{n+1}^{\infty} a_\nu^2 \right)^{\frac{1}{2}}. \tag{35}$$

Now, if

$$\sum_{n=1}^{\infty} \lambda_n^{\frac{3}{2}} a_n^2 < \infty \tag{36}$$

then

$$\sum_{\nu=1}^{n} \lambda_\nu^2 a_\nu^2 = o\,(\lambda_n^{\frac{1}{2}}), \tag{37}$$

and as in (29)

$$\sum_{\nu=n+1}^{\infty} a_\nu^2 = o\,(\lambda_n^{-\frac{3}{2}}). \tag{38}$$

Hence, we have in place of (30)

$$|f - S_n| \leqslant |f - M_R[f]| + o\left(\frac{1}{R\lambda_n^{\frac{1}{2}}} \right)^{\frac{3}{2}} + o\left(R\lambda_n^{\frac{1}{2}} \right)^{\frac{1}{2}}. \tag{39}$$

Choosing $R = \lambda_n^{-\frac{1}{2}}$ and letting $n \to \infty$, we prove Theorem 2.

CONCLUDING REMARKS. Theorems 1 and 2 remain true even in the cases of the second and third boundary value problems, as the inequalities (14) and (33) continue to be true even then. In fact, these continue to be true even in the case where $\omega_n(x, y)$ are the eigenfunctions of a symmetric kernel $K(x, y; \xi, \eta)$ of the Green's function type defined in remark (iv) of the first paper.

ON FUNCTIONS WHICH ARE HANKEL-TRANSFORMS OF EACH OTHER

BY

HARI SHANKER, *Anglo-Arabic College, Delhi*

[Received 30 January 1941]

1. The object of this paper is to establish the following

THEOREM I. *If* $\phi(x) = m!\, 2^m e^{-\frac{1}{2}x^2} x^{2n+\nu+\frac{1}{2}} L_m^{n+\nu}(\tfrac{1}{2}x^2)$

and $\psi(x) = n!\, 2^n e^{-\frac{1}{2}x^2} x^{2m+\nu+\frac{1}{2}} L_n^{m+\nu}(\tfrac{1}{2}x^2)$

then $\phi(x)$ *and* $\psi(x)$ *are Hankel-transforms of each other of order* ν, *provided that* m *and* n *are positive integers.*

2. For this purpose we use Varma's integral*

$$\int_0^\infty e^{-\frac{1}{4}t^2} t^{2l-\nu-1} J_\nu(xt)\, W_{k,\mu}(\tfrac{1}{2}t^2)\, dt$$

$$= \frac{2^{l-\nu-1} x^\nu \Gamma(l+\mu+\frac{1}{2})\Gamma(l-\mu+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(l-k+1)}\, {}_2F_2\left[\begin{matrix} l+\mu+\frac{1}{2},\ l-\mu+\frac{1}{2}; \\ \nu+1,\ l-k+1; \end{matrix} -\tfrac{1}{2}x^2\right],$$

$$l \pm \mu + \tfrac{1}{2} > 0. \qquad (2.1)$$

Taking $2\mu = m+\nu$, $k = 2n+m+\nu+1$, and $2l = m+\nu+1$, and noting that

$$e^{\frac{1}{4}t^2} W_{\frac{2n+m+\nu+1}{2},\, \frac{m+\nu}{2}}(\tfrac{1}{2}t^2) = n!\,(-1)^n 2^{-\left(\frac{m+\nu+1}{2}\right)} t^{m+\nu+1} L_n^{m+\nu}(\tfrac{1}{2}t^2),$$

we get after a little rearrangement

$$n!\,(-1)^n \int_0^\infty e^{-\frac{t^2}{2}} t^{2m+\nu+\frac{1}{2}} L_n^{m+\nu}(\tfrac{1}{2}t^2) J_\nu(xt)\,(xt)^{\frac{1}{2}} dt$$

$$= \frac{2^m x^{\nu+\frac{1}{2}} \Gamma(m+\nu+1)}{\Gamma(\nu+1)\Gamma(1-n)}\, {}_2F_2\left[\begin{matrix} m+\nu+1,\ 1; \\ \nu+1,\ 1-n; \end{matrix} -\tfrac{1}{2}x^2\right]$$

$$= 2^m x^{\nu+\frac{1}{2}} \sum_{r=0}^\infty \frac{\Gamma(m+\nu+r+1)}{\Gamma(\nu+r+1)\Gamma(1-n+r)} (-\tfrac{1}{2}x^2)^r$$

* R. S. Varma: *Jour. Ind. Math. Soc.* (2) 3 (1938), 25-33.

$$= 2^m x^{\nu+\frac{1}{2}} \sum_{r=n}^{\infty} \frac{\Gamma(m+\nu+r+1)}{\Gamma(\nu+r+1)\Gamma(1-n+r)} (-\tfrac{1}{2}x^2)^r$$

since $\Gamma(1-n+r) = \infty$ when $r < n$,

$$= \frac{(-1)^n 2^{m-n} x^{2n+\nu+\frac{1}{2}} \Gamma(m+\nu+n+1)}{\Gamma(\nu+n+1)}$$
$$\times {}_1F_1[m+\nu+n+1;\ n+\nu+1;\ -\tfrac{1}{2}x^2]$$

$$= \frac{(-1)^n 2^{m-n} \Gamma(m+\nu+n+1) e^{-\frac{1}{2}x^2} x^{2n+\nu+\frac{1}{2}}}{\Gamma(\nu+n+1)}$$
$$\times {}_1F_1[-m;\ n+\nu+1;\ \tfrac{1}{2}x^2], \qquad (2.2)$$

by virtue of Kummer's first transformation formula

$${}_1F_1(a;\ b;\ x) = e^x {}_1F_1(b-a;\ b;\ -x).$$

Now since $L_\mu^\nu(z) = \dfrac{\Gamma(\mu+\nu+1)}{\mu!\ \Gamma(\nu+1)}\, {}_1F_1[-\mu;\ \nu+1,\ z]$,

the relation (2.2) can be expressed in the form

$$n!\ 2^n \int_0^\infty e^{-\frac{t^2}{2}} t^{2m+\nu+\frac{1}{2}} L_n^{m+\nu}(\tfrac{1}{2}t^2) J_\nu(xt) \sqrt{xt}\,dt$$
$$= m!\ 2^m e^{-\frac{1}{2}x^2} x^{2n+\nu+\frac{1}{2}} L_m^{n+\nu}(\tfrac{1}{2}x^2), \qquad (2.3)$$

which is the same as $\phi(x) = \int_0^\infty \psi(t) J_\nu(xt)(xt)^{\frac{1}{2}} dt.$ (2.4)

To prove the reciprocal relation

$$\psi(x) = \int_0^\infty \phi(t) J_\nu(xt)(xt)^{\frac{1}{2}} dt, \qquad (2.5)$$

we simply interchange n and m in (2.3) and obtain

$$n!\ 2^n e^{-\frac{1}{2}x^2} x^{2m+\nu+\frac{1}{2}} L_n^{m+\nu}(\tfrac{1}{2}x^2)$$
$$= m!\ 2^m \int_0^\infty e^{-t^2/2} t^{2n+\nu+\frac{1}{2}} L_m^{n+\nu}(\tfrac{1}{2}t^2) J_\nu(xt)(xt)^{\frac{1}{2}} dt, \qquad (2.6)$$

which proves the theorem.

3. The following special cases of this theorem are worth mentioning.

CASE I. When $n = m$ we obtain Howell's result*, viz. the function

$$\phi(x) = e^{-\frac{1}{2}x^2} x^{2n+\nu+\frac{1}{2}} L_n^{n+\nu}(\tfrac{1}{2}x^2) \text{ is } R_\nu, \qquad (3.1)$$

i.e. self-reciprocal in the Hankel-transform of order ν.

* W. T. Howell, *Phil. Mag.* (7) 23 (1937), 807-11.

CASE 2. Putting $m=0$ we have

$$\phi(x)=e^{-\frac{1}{2}x^2}x^{2n+\nu+\frac{1}{2}} \text{ and } \psi(x)=n!\,2^n e^{-\frac{1}{2}x^2}x^{\nu+\frac{1}{2}}L_n^{\nu}(\tfrac{1}{2}x^2) \tag{3.2}$$

as a pair of Hankel-transforms.

CASE 3. Putting $\nu=\frac{1}{2}$, we find that

$$\phi(x)=2^m m!\,e^{-\frac{1}{2}x^2}x^{2n+1}L_m^{n+\frac{1}{2}}(\tfrac{1}{2}x^2)$$

and

$$\psi(x)=2^n n!\,e^{-\frac{1}{2}x^2}x^{2m+1}L_n^{m+\frac{1}{2}}(\tfrac{1}{2}x^2)$$

are Fourier sine-transforms of each other. (3.3)

CASE 4. Similarly, putting $\nu=-\frac{1}{2}$, we get

$$\phi(x)=2^m m!\,e^{-\frac{1}{2}x^2}x^{2n}L_m^{n-\frac{1}{2}}(\tfrac{1}{2}x^2)$$

and

$$\psi(x)=2^n n!\,e^{-\frac{1}{2}x^2}x^{2m}L_n^{m-\frac{1}{2}}(\tfrac{1}{2}x^2)$$

as a pair of Fourier cosine-transforms. (3.4)

Lastly we conclude this section by deducing a simple proof from Theorem 1 of Erdelyi's formula*, viz.

$$z^m L_n^{m+\nu}(z)=\frac{\Gamma(m+n+\nu+1)}{n!}\sum_{r=0}^{m}\frac{(-1)^r\,{}_mC_r\,(n+r)!\,L_{n+r}^{\nu}(z)}{\Gamma(n+\nu+r+1)}. \tag{3.5}$$

Replacing $L_m^{n+\nu}(\frac{1}{2}t^2)$ in (2.6) by the well-known expression

$$L_m^{n+\nu}(\tfrac{1}{2}t^2)=\Gamma(m+n+\nu+1)\sum_{r=0}^{m}\frac{(-\frac{1}{2}t^2)^r}{(m-r)!\,r!\,\Gamma(n+\nu+r+1)}$$

and integrating term by term with the aid of (3.2), we obtain

$$2^n n!\,e^{-\frac{1}{2}x^2}x^{2m+\nu+\frac{1}{2}}L_n^{m+\nu}(\tfrac{1}{2}x^2)$$

$$=2^{m+n}m!\,\Gamma(m+n+\nu+1)\sum_{0}^{m}\frac{e^{-\frac{1}{2}x^2}x^{\nu+\frac{1}{2}}(-1)^r(n+r)!\,L_{n+r}^{\nu}(\frac{1}{2}x^2)}{(m-r)!\,r!\,\Gamma(n+\nu+r+1)},$$

which after a little simplification is found to be the same as (3.5) with z replaced by $x^2/2$.

4. In conclusion we take the opportunity of noting that term by term computation, by Theorem 1, of the Hankel-transform of any function $f(x)$, where $f(x)$ is the sum of any finite number of terms of the type $\phi(x)$ yields another series of terms, finite in number and of the type

* A. Erdēlyi, *Jour. Lond. Math. Soc.* 13 (1938), 154-6.

$\psi(x)$, and that if $F(x)$ be the sum of this series, then $f(x)$ and $F(x)$ are Hankel-transforms of each other. This may be well illustrated by taking for example

$$f(x)=(-1)^n 2^m e^{-\frac{1}{2}x^2} x^{\nu+\frac{1}{2}} \sum_{r=0}^{n} \frac{(m+n-r)!}{(n-r)!\,r!} \frac{(-x^2)^r}{2^r} L_{n+m-r}^{\nu+r}(\tfrac{1}{2}x^2).$$

Hence computing $F(x)=\int_0^\infty f(t) J_\nu(xt)(xt)^{\frac{1}{2}}dt$ (4·1)

term by term with the help of Theorem 1, we have, after arrangement

$$F(x) = e^{-\frac{1}{2}x^2} x^{2m+\nu+\frac{1}{2}} \sum_{r=0}^{n} \frac{L_{n-r}^{m+\nu+r}(\frac{1}{2}x^2)\,(-x^2)^r}{r!\,2^r}. \qquad (4·2)$$

Similarly the reciprocal formula

$$f(x)=\int_0^\infty F(t) J_\nu(xt)(xt)^{\frac{1}{2}}dt \qquad (4·3)$$

can be easily established. In particular, when $m=0$, $f(x)=(-1)^n F(x)$ and we obtain from (4·1) the following

THEOREM 2. *The function*

$$F(x)=e^{-\frac{1}{2}x^2} x^{\nu+\frac{1}{2}} \sum_{r=0}^{n} \frac{L_{n-r}^{\nu+r}(\frac{1}{2}x^2)\,(-x^2)^r}{r!\,2^r}$$

is self-reciprocal or skew-reciprocal in the Hankel-transform of order ν *according as* n *is even or odd.* (4·4)

Since it is not difficult to prove that $F(x)$ as defined above is equal to $e^{-\frac{1}{2}x^2} x^{\nu+\frac{1}{2}} L_n^\nu(x^2)$, we find that (4·4) is the same as Wilson's theorem*, viz.

The function

$$F(x) = e^{-\frac{1}{2}x^2} x^{\nu+\frac{1}{2}} L_n^\nu(\tfrac{1}{2}x^2) \text{ is } R_\nu \text{ or } -R_\nu$$

according as n *is even or odd.*

* B. M. Wilson, *Messenger of Math.* 53 (1923), 157-60.

POSTSCRIPT TO "CONVERGENCE THEOREMS FOR SERIES OF POSITIVE TERMS"*

BY

C. T. RAJAGOPAL, *Madras Christian College*

[Received 5 February 1941]

1. In the paper cited above, I have based the convergence theory of positive series on

THEOREM A. *If* $0 = d_0 \equiv D_0$, $0 < d_n \equiv D_n - D_{n-1}$ $(n = 1, 2, \ldots)$, $D_n \to \infty$, *and if* $F(x)$ *is positive monotone decreasing,* $\int^{\infty} F(x)\,dx$ *is convergent, then* $\sum d_n F(D_n)$ *is convergent.*

The first result of this note leads to the more general companion

THEOREM B. *If* $0 = e_0 \equiv E_0$, $e_n \equiv E_n - E_{n-1}$, $(n = 1, 2, \ldots)$, $E_n = O(D_n)$; *then under the hypotheses of Theorem A,* $\Sigma e_n F(D_n)$ *is convergent.*

This carries with it the

COROLLARY. *Under the hypotheses of Theorem A,* $\Sigma(-)^{n+1} D_n F(D_n)$ *is convergent.*

When $d_n = 1$, Theorem B and its corollary reduce to results essentially equivalent to Satz 20 and Satz 21 of Th. Kaluza's paper "Struktur und Eigenschaften mehrfach monotoner Folgen" in *Math. Zeit.* 26 (1927), 345-64.

The second theorem of this note is a completion of Theorem 4 of my paper referred to in the title.

1.1. The following simple lemmas are required in our proofs.

* *Jour. Ind. Math. Soc.* (2), 3 (1938), 118-25.

LEMMA 1. *If the sequence* $\{a_n/d_n\}$ *is positive monotone decreasing and the series* Σa_n *is convergent, then* $a_n D_n/d_n \to 0$.*

LEMMA 2. *Under the hypothesis of Lemma 1,*

$$\sum_{n=1}^{\infty} D_n\left(\frac{a_n}{d_n}-\frac{a_{n+1}}{d_{n+1}}\right)=\sum_{n=1}^{\infty} a_n \dagger.$$

2. THEOREM 1. *If* $\{E_n\}$ *is as in Theorem B and if* $\{a_n/d_n\}$, Σa_n *are as in Lemma 1, then* $\Sigma e_n a_n/d_n$ *is convergent.*

PROOF. Lemma 2 together with the restriction on E_n shews that $\sum E_n\left(\frac{a_n}{d_n}-\frac{a_{n+1}}{d_{n+1}}\right)$ is absolutely convergent. Now

$$\sum_{\nu=1}^{n} E_\nu\left(\frac{a_\nu}{d_\nu}-\frac{a_{\nu+1}}{d_{\nu+1}}\right)=\sum_{\nu=1}^{n}(E_\nu-E_{\nu-1})\frac{a_\nu}{d_\nu}-E_n\frac{a_{n+1}}{d_{n+1}},$$

where

$$|E_n a_{n+1}/d_{n+1}| \leqslant KD_n a_n/d_n \to 0,$$

in consequence of Lemma 1. Hence follows the convergence of

$$\sum(E_n-E_{n-1})a_n/d_n=\sum e_n a_n/d_n.$$

Taking $E_n=(-1)^{n+1}D_n/2$, $(n \geqslant 1)$, we find

$$\sum_{\nu=1}^{n}(E_\nu-E_{\nu-1})\frac{a_\nu}{d_\nu}=\sum_{\nu=1}^{n}(-)^{\nu+1}D_\nu\frac{a_\nu}{d_\nu}+\frac{1}{2}\sum_{\nu=1}^{n}(-1)^\nu a_\nu$$

whence follows the

COROLLARY. *If* $\{a_n/d_n\}$ *and* Σa_n *are as in Lemma 1, then* $\Sigma(-1)^{n+1}D_n a_n/d_n$ *is convergent.*

REMARKS. (i) If $F(x)$ is as in Theorem A, we may take $a_n=d_nF(D_n)$ in the theorem and in the corollary and restate these as Theorem B and its corollary.

* "On Abel's divergence test for series of positive terms", *Math. Student*, 8 (1940), 118-23, Theorem 1.

† *Ibid.* Theorem 2 (*a*).

(ii) In the corollary, we can assert the convergence of $\Sigma(-1)^{n+1}D'_n a_n/d_n$ when $\{D'_n\}$ is a strictly increasing divergent sequence such that $(D'_n-D'_{n-1}) \sim k(D_n-D_{n-1})$. Thus, with $d_n = 1/n^{1-\alpha}$ $(\alpha > 0)$ and $D'_n = n^\alpha$, $k = \alpha$. Writing $a_n = b_n/n$ we can now state the corollary in the form:

If, for $\alpha > 0$, $\{b_n/n^\alpha\}$ is positive monotone decreasing, and $\Sigma b_n/n$ is convergent, then $\Sigma(-1)^{n+1}b_n$ is convergent.

This is Satz 23 of Th. Kaluza's paper where it is proved by considerations involving sequences "α-fach monoton". The proof of the satz given here shows that it is one of a general class to which results such as the following also belong. (In these results, $l_1 n$, $l_2 n, \ldots$ denote respectively log n, log log $n, \ldots$)

If, for $\alpha > 0$, $\left\{\dfrac{b_n l_1 n l_2 n \ldots l_p n}{(l_p n)^\alpha}\right\}$ is positive monotone decreasing and $\Sigma b_n/n$ is convergent, then $\Sigma(-1)^{n+1}b_n l_1 n l_2 n \ldots l_p n$ is convergent.

If $\{b_n l_1 n l_2 n \ldots l_p n\}$ is positive monotone decreasing, and $\Sigma b_n/n$ is convergent, then $\Sigma(-1)^{n+1}b_n l_1 n l_2 n \ldots l_{p+1} n$ is convergent. In particular,

If $\{b_n\}$ is positive monotone decreasing, and $\Sigma b_n/n$ is convergent, then $\Sigma(-1)^{n+1}b_n l_1 n$ is convergent.

3. THEOREM 2. *If Σa_n and Σd_n are series of positive terms of which the latter is divergent (i.e. $D_n = \sum_{\nu=1}^{n} d_\nu \to \infty$), and if $-f(x)$ is positive monotone decreasing, satisfying the conditions:*

$$\frac{1}{d_{n+1}}\log\frac{a_{n+1}d_n}{a_n d_{n+1}} \leqslant f(D_n), \quad \int^\infty \exp\left\{\int^x f(t)\,dt\right\}dx \text{ is convergent,}$$

then Σa_n and $\Sigma(-1)^{n+1}a_n D_n/d_n$ are convergent.

PROOF. The first conclusion, already proved,* when taken in conjunction with the corollary under Theorem 1, leads to the second.

The theorem shews that Kummer's convergence criterion $[f(x)=\rho<0]$ and the generalized de Morgan-Bertrand convergence criteria

$$\left[f(x)=-\frac{1}{x}-\frac{1}{xl_1x}-\cdots-\frac{1+\rho}{xl_1x\ldots l_px},\ (\rho>0)\right]$$

establish at one stroke the convergence of

$$\Sigma a_n \text{ and } \Sigma(-1)^{n+1}a_nD_n/d_n.$$

As an example of Theorem 2, consider

$$a_{n+1}=d_{n+1}\frac{\prod_{\nu=0}^{n}(\alpha d_{\nu+1}+D_\nu)(\beta d_{\nu+1}+D_\nu)}{\prod_{\nu=0}^{n}(\gamma d_{\nu+1}+D_\nu)(\delta d_{\nu+1}+D_\nu)},$$

where $d_n=O(1)$, and $\alpha, \beta, \gamma, \delta$ are real, the two latter $\neq -D_\nu/d_{\nu+1}$, $(\nu=0, 1, 2, \ldots)$. In this case,

$$\frac{a_n/d_n}{a_{n+1}/d_{n+1}}=1+(\gamma+\delta-\alpha-\beta)\frac{d_{n+1}}{D_n}+O\left(\frac{d_{n+1}}{D_n^2}\right),$$

or,

$$\frac{1}{d_{n+1}}\log\frac{a_nd_{n+1}}{a_{n+1}d_n}=-\frac{\gamma+\delta-\alpha-\beta}{D_n}+O\left(\frac{1}{D_n^2}\right).$$

Hence Theorem 2 establishes the convergence of $\Sigma(-1)^{n+1}a_nD_n/d_n$ when $\gamma+\delta-\alpha-\beta>1$.†

* *Jour. Ind. Math. Soc.* loc. cit. 123-4.

† For the convergence of $\Sigma(-1)^{n+1}a_n/d_n$, when $\gamma+\delta-\alpha-\beta>0$, see "Remarks on some generalizations of Cauchy's condensation and integral tests", *Amer. Math. Monthly*, 48 (1941), 180-5.

ON WARING'S PROBLEM FOR SMALLER POWERS

BY

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[Received in revised form, 2 March 1941]

In a manuscript, L. K. Hua has proved, among other things, that

$$G(5) \leqslant 24, G(6) \leqslant 37 \text{ and } G(7) \leqslant 53,$$

these being improvements on my results*. Before the receipt of Hua's manuscript, I had pointed out that $G(5) \leqslant 24$. The main lemma† of Hua's argument is aimed at improving the lower limits for $H_{s,k}(N)$, the number of integers less than N and expressible as the sum of s positive kth powers. The purpose of the present paper is to improve the main lemma of Hua's argument and consequently to prove

THEOREM 1.

$$G(5) \leqslant 24, \; G(6) \leqslant 36 \textit{ and } G(7) \leqslant 52.$$

The same principle is applicable to other cases of $G(k)$ as well.

Hua's theory is just enough to prove that every sufficiently large integer is expressible as the sum of 24 positive fifth powers, whereas the results obtained here enable us to prove in addition

THEOREM 2. *Every sufficiently large integer is expressible as the sum of 23 fifth powers and a sixth power.*

Some applications of Lemma 1 of this paper to the Additive Prime Number Theory are the following:

* *Math. Student*, 7 (1939), 144, 169.

† Due to Davenport as given in Hua's manuscript.

Let p denote an odd prime and $\psi_1(x)$, $\psi_2(x), \ldots$, $\psi_r(x)$, $(r \geqslant 1)$ denote any integral-valued polynomials whose first coefficients are all positive. For the sake of definiteness, we shall suppose that $\psi_1(1)\psi_2(1)\ldots\psi_r(1)$ is odd. Then we have

THEOREM 3. *Every sufficiently large odd integer can be represented in any one of the three forms:*

$$(1)\ n = p_1^5 + p_2^5 + \cdots + p_{24}^5 + \psi_1(p_{25})\cdots\psi_r(p_{r+24}),$$

$$(2)\ n = p_1^6 + p_2^6 + \cdots + p_{36}^6 + \psi_1(p_{37})\cdots\psi_r(p_{r+36}),$$

provided $n \equiv \psi_1(x_1)\ldots\psi_r(x_r) \pmod{9}$, *where* $x_1, \ldots, x_r$ *may take values prime to* 9,

$$(3)\ n = p_1^7 + p_2^7 + \cdots + p_{52}^7 + \psi_1(p_{53})\cdots\psi_r(p_{r+52}).$$

The proof of this theorem is similar to that of the theorems stated in a previous paper*.

We require the following lemmas for the proof of Theorem 1.

LEMMA† 1. *Let* $\{M\}$ *denote a set of distinct natural numbers satisfying the following conditions:*

(*a*) *the number of elements of* $\{M\}$ *is* U,

and (*b*) *each lement of* $\{M\}$ *is* $\leqslant P^\delta$.

Then the number N_0 *of solutions of*

$$f(x_1) + M_1 = f(x_2) + M_2,\ P \leqslant x_1, x_2 \leqslant 2P;\ M_1, M_2 \,\varepsilon\, \{M\}, \quad (1)$$

is

$$= O\left\{ P^{1+\varepsilon} U (1 + P^{\delta - k + 1 - 2^{-r+1}} + P^{(\delta-k-1)(1-2^{-r}) - (r+1)2^{-r}} U^{-2^{-r}} U_{\vdots}^{2^{1-k}} \right\}$$

for $1 \leqslant r \leqslant k-2$, *where* ε *denotes an arbitrary small positive number.*

* Rao, Generalizations of a theorem of Estermann in the additive prime-number theory, *Dr. C. R. Reddi Sashtipoorthi Volume, Andhra University* (1940).

† *Cf.* Davenport: On Waring's Problem for fourth powers, Lemma 1, *Annals of Math.* 40 (1939), 731-47.

PROOF. Let $f(x)$ denote a polynomial of the kth degree with integer coefficients and $t_1\Delta f(x)=f(x+t_1)-f(x)$; $t_1t_2\Delta^2 f(x)=t_1[\Delta f(x+t_2)-\Delta f(x)]$; and so on. The number of solutions of (1) with $x_1=x_2$ is PU. To find the number of solutions of (1) with $x_1>x_2$, put $x_2=x$ and $x_1=x+t_1$. Then (1) becomes

$$t_1\Delta f(x)+M_1=M_2, \tag{2}$$

where

$$t_1<P^{\delta-k+1}.$$

Let N_r denote the number of solutions of

$$t_1\ldots t_r\Delta^r f(x)+M_1=M_2, \tag{3}$$

where

$$t_1\ldots t_r=O(P^{\delta-k+r}).$$

Let $r(M_2, t)$ denote the number of solutions of (3) for given values of M_2 and t. Then by Cauchy's inequality we have

$$N_r=\sum_t\sum_{M_2}r(M_2, t)\leqslant\Big(\sum_t\sum_{M_2}1\Big)^{\frac{1}{2}}\Big(\sum_t\sum_{M_2}r^2[M_2, t]\Big)^{\frac{1}{2}}$$

$$\leqslant(P^{\delta-k+r}U)^{\frac{1}{2}}N'^{\frac{1}{2}}_r, \tag{4}$$

where N'_r is the number of solutions of

$$t_1\ldots t_r\Delta^r f(x)+M_1=t_1\ldots t_r\Delta^r f(y)+M'_1=M_2. \tag{5}$$

But the number of solutions of (5) with $x=y$ is precisely N_r. Hence to find the number of solutions with $y>x$ put $y=x+t_{r+1}$. Then the number of solutions of (5) is exactly N_{r+1}. Hence

$$N'_r=\text{Max }(N_r, N_{r+1}).$$

Hence from (4) we have

$$N_r<<\text{Max }\{P^{\delta-k+r}U,\ P^{\frac{1}{2}(\delta-k+r)}U^{\frac{1}{2}}N^{\frac{1}{2}}_{r+1}\}. \tag{6}$$

Also, for given values of M_1 and M_2 and for $1\leqslant r\leqslant k-1$ the number of solutions for $t_1, t_2, \ldots, t_r$ and $\Delta^r f(x)$ as factors of M_2-M_1 is $<<d(M_2-M_1)$ and hence is $O(P^\varepsilon)$.

Hence for
$$1 \leqslant r \leqslant k-1$$
$$\mathcal{N}_r = O(U^2 P^{\varepsilon}).$$

Now we shall prove that

$$\mathcal{N}_1 << UP^{\delta-k+1+1-2^{-r+1}} + U^{1-2^{-r}} P^{(\delta-k+1)(1-2^{-r})+1-(r+1)2^{-r}} \mathcal{N}_{r+1}^{2^{-r}}. \qquad (8)$$

This can easily be proved by induction. (8) follows from (6) for $r=1$. Let us assume that (8) is true for r. Then from (6) and (8) we have

$$\mathcal{N}_1 << UP^{\delta-k+1+1-2^{-r+1}} + U^{1-2^{-r}} P^{(\delta-k+1)(1-2^{-r})+1-(r+1)2^{-r}} \times \left\{ P^{\delta-k+r+1} U + P^{\frac{1}{2}(\delta-k+r+1)} U^{\frac{1}{2}} \mathcal{N}_{r+2}^{\frac{1}{2}} \right\}^{2^{-r}}$$
$$<< UP^{\delta-k+1+1-2^{-r}} + U^{1-2^{-r}} P^{(\delta-k+1)(1-2^{-r})+1-(r+2)2^{-r-1}} \mathcal{N}_{r+2}^{2^{-r-1}},$$

which shows that (8) is true for $r+1$ when it is true for r. Hence (8) is true for $1 \leqslant r \leqslant k-2$. Since

$$\mathcal{N}_0 << PU + \mathcal{N}_1,$$

from (7) and (8) we have the lemma.

In what follows ε denotes an arbitrarily small positive number and P is a sufficiently large positive number.

Suppose that we are given a set of integers $\leqslant P^k$ which are expressible as the sum of $(s-1)$ kth powers. Let $P^{X_{s-1}-\varepsilon}$ denote the number of elements of the set. Then we have

LEMMA 2. *There exists a set of integers $\leqslant P^k$ which are expressible as the sum of s kth powers and is such that, if $P^{X_s-\varepsilon}$ denotes the number of its elements, then*

$$X_s = 1 + \lambda X_{s-1},$$

where λ satisfies the two inequalities

$$\lambda \leqslant (k-1+2^{-r+1})/k$$
$$\lambda \leqslant \left\{ (k-1)(1-2^{-r}) + (r+1)2^{-r} \right\} / \left\{ k(1-2^{-r}) + 2^{-r} X_{s-1} \right\},$$

where r is an integer such that $1 \leqslant r \leqslant k-2$.

PROOF. This lemma follows from Lemma 1 when we take $\delta = k\lambda$ and note that $U << P^{\lambda X_{s-1}}$.

From this lemma it follows that for any r for which

$$X_{s-1} > k\frac{r-1+2^{-r+1}}{k-1+2^{-r+1}}$$

we can take

$$\lambda = \frac{(k-1)(1-2^{-r})+(r+1)2^{-r}}{k(1-2^{-r})+2^{-r}X_{s-1}}. \tag{9}$$

LEMMA 3. *The number of integers* $\leqslant P^5$ *and expressible as the sum of* 8 *fifth powers is greater than*

$$P^{4.55534-\varepsilon}$$

LEMMA 4. *The number of integers* $\leqslant P^6$ *and expressible as the sum of* 13 *sixth powers is greater than*

$$P^{5.68943\ldots-\varepsilon}.$$

LEMMA 5. *The number of integers* $\leqslant P^7$ *which are expressible as the sum of* 19 *seventh powers is greater than*

$$P^{6.78163\ldots-\varepsilon}.$$

Lemmas 3, 4 and 5 follow from Lemma 2 and (9).

Theorem 1 can now be proved in the usual manner by using Lemmas 3, 4 and 5.

In conclusion I wish to thank the referee for his helpful criticism.

Note added on 10-5-41 :—In a recent letter Dr. Hua mentions that Mr. Davenport has also obtained the above results for $G(5)$ and $G(6)$.

ON WARING'S PROBLEM FOR 8TH, 9TH, AND 10TH POWERS

BY

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1. T. Estermann's result* gives us $G(k) \leqslant s$, where
$s = 2m_0+7+[2^{k-1}(k-2)(1-k^{-1})^{m_0+1}]$ and

$$m_0=\left[\frac{(k-2)\log 2+\log (k-2)-\log k}{\log k-\log (k-1)}\right],$$

the expression $[x]$ meaning the integral part of x. When $k=8$, 9 and 10, the above result gives us

$$G(8) \leqslant 78,\ G(9) \leqslant 101, \text{ and } G(10) \leqslant 125.$$

In the present paper I calculate the values of $G(8)$, $G(9)$, and $G(10)$ by means of Rao's method, explained in the preceding paper. The results of calculation can be given as

THEOREM. $G(8) \leqslant 73$ (*A*)

$G(9) \leqslant 99$ (*B*)

and $G(10) \leqslant 122$. (*C*)

It may be noted that, as k increases, the values of $G(k)$ obtainable by this method will not be very much different from those of Estermann's.

2. Let $H_{s,k}(N)$ denote the number of integers $\leqslant N$ that are expressible as the sum of s positive integral kth powers. Following Rao's method, on calculation, we find

$$H_{28,8}(P^8) > P^{7\cdot 87115-\varepsilon} \tag{1}$$

$$H_{40,9}(P^9) > P^{8\cdot 92611-\varepsilon} \tag{2}$$

and

$$H_{51,10}(P^{10}) > P^{9\cdot 96128-\varepsilon}. \tag{3}$$

Now (*A*), (*B*) and (*C*) follow from (1), (2) and (3) respectively, in the usual way.

* On Waring's problem for 4th and higher powers, *Acta Arithmetica*, 2 (1937).

129112

SOME INFINITE INTEGRALS

BY

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1. In a recent paper* I have studied the properties of the function

$$\phi_\nu(x) = \sum_0^\infty \frac{(-1)^r \Gamma(\frac{1}{2}+r) x^{\nu+2r}}{2^{\nu+2r}\Gamma(1+\nu+r)\, r!}, \quad R(\nu) > -1.$$

I have shown therein that

$$\phi_\nu(x) = \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^1 u^{-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}} e^{-\frac{1}{4}x^2 u}\, du. \qquad (1.1)$$

The object of this note is to evaluate a few infinite integrals involving this function.

To begin with, we have

$$\phi_\nu(x) = \frac{x^{\nu-1}}{2^\nu \Gamma(\nu+\frac{1}{2})}\int_0^{x^2} t^{-\frac{1}{2}}\left(1-\frac{t}{x^2}\right)^{\nu-\frac{1}{2}} e^{-\frac{1}{4}ut}\, dt$$

$$\sim x^{\nu-1} \text{ for large } x.$$

2. We proceed to evaluate the integral

$$I = \int_0^\infty x^{p-1}\phi_\nu(ax)\,dx, \quad R(a)>0,\ 0 < R(\nu+p) < 1$$

$$= \frac{a^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty x^{p+\nu-1}dx\int_0^1 u^{-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}e^{-\frac{1}{4}a^2x^2u}du,$$

on using (1.1). Thus

$$I = \frac{a^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^1 u^{-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}du\int_0^\infty x^{p+\nu-1}e^{-\frac{1}{4}a^2ux^2}dx,$$

a process easily justifiable,

$$= \frac{2^{p-1}\Gamma(\frac{1}{2}p+\frac{1}{2}\nu)}{a^p\Gamma(\nu+\frac{1}{2})}\int_0^1 u^{-\frac{1}{2}p-\frac{1}{2}\nu-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}du.$$

* A certain confluent hypergeometric function—in the press.

Thus, we have

$$\int_0^\infty x^{p-1}\phi_\nu(ax)\,dx = \frac{2^{p-1}}{a^p}\frac{\Gamma(\frac{1}{2}p+\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}-\frac{1}{2}p-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}p+\frac{1}{2}\nu)}, \qquad (2.1)$$

$$R(a)>0,\ 0<R(\nu+p)<1,\ R(\nu)>-1.$$

PARTICULAR CASES.

(i) $p=1$

$$a\int_0^\infty \phi_\nu(ax)\,dx = \Gamma(-\tfrac{1}{2}\nu), \quad -1<R(\nu)<0. \qquad (2.2)$$

For $\nu=-\frac{1}{2}$ this reduces to the well-known result

$$\int_0^\infty x^{-\frac{1}{2}}e^{-\frac{1}{4}a^2x^2}dx = (2a)^{-\frac{1}{2}}\Gamma(\tfrac{1}{4}),$$

on using the formula

$$\phi_{-\frac{1}{2}}(x) = \left(\frac{2}{x}\right)^{\frac{1}{2}}e^{-\frac{1}{4}x^2}.$$

(ii) $p=\nu$

$$\int_0^\infty x^{\nu-1}\phi_\nu(ax)\,dx = \frac{2^{\nu-1}}{a^\nu}\Gamma(\nu)\Gamma(\tfrac{1}{2}-\nu), \quad 0<R(\nu)<\tfrac{1}{2}. \qquad (2.3)$$

(iii) $p=\nu-2$

$$\int_0^\infty x^{\nu-3}\phi_\nu(ax)\,dx = \frac{2^{\nu-3}}{a^{\nu-2}}\Gamma(\nu-1)\Gamma(\tfrac{3}{2}-\nu), \quad 1<R(\nu)<\tfrac{3}{2} \qquad (2.4)$$

(iv) $\nu=-\frac{1}{2}$

$$\int_0^\infty x^{p-\frac{3}{2}}e^{-\frac{1}{4}a^2x^2}dx = \frac{2^{p-\frac{3}{2}}}{a^{p-\frac{1}{2}}}\Gamma(\tfrac{1}{2}p-\tfrac{1}{4}), \quad \tfrac{1}{2}<R(p)<\tfrac{3}{2},$$

which is a familiar result.

3. The integral

$$K=\int_0^\infty x^{-p}\phi_{p+\nu+1}(ax)J_\nu(bx)\,dx, \quad R(a)>0,\ R(b)>0,$$

$$R(p+\nu)\geqslant -\tfrac{3}{2}$$

is convergent, by the theory of analytic continuation, for $-1<R(\nu)<\frac{1}{2}$. We have, by (1.1),

$$K=\frac{a^{p+\nu+1}}{2^{p+\nu+1}\Gamma(p+\nu+\frac{3}{2})}$$

$$\times\int_0^\infty x^{\nu+1}J_\nu(bx)\,dx\int_0^1 u^{-\frac{1}{2}}(1-u)^{p+\nu+\frac{1}{2}}e^{-\frac{1}{4}a^2x^2u}du$$

$$=\frac{a^{p+\nu+1}}{2^{p+\nu+1}\Gamma(p+\nu+\frac{3}{2})}$$
$$\times\int_0^1 u^{-\frac{1}{2}}(1-u)^{p+\nu+\frac{1}{2}}du\int_0^\infty x^{\nu+1}e^{-\frac{1}{4}a^2ux^2}J_\nu(bx)\,dx,$$

a process easily justifiable. Thus, using Sonine's formula* we have

$$K=\frac{a^{p-\nu-1}b^\nu}{2^p\Gamma(p+\nu+\frac{3}{2})}\int_0^1\frac{(1-u)^{p+\nu+\frac{1}{2}}}{u^{\nu+\frac{3}{2}}}e^{-\frac{b^2}{a^2u}}du$$
$$=\frac{a^{p-\nu-1}b^\nu}{2^p\Gamma(p+\nu+\frac{3}{2})}e^{-\frac{b^2}{a^2}}\int_0^\infty\frac{t^{p+\nu+\frac{1}{2}}e^{-\frac{b^2}{a^2}t}}{(1+t)^{p+1}}dt.$$

On using a formula given by Whittaker† we get

$$\int_0^\infty x^{-p}\phi_{p+\nu+1}(ax)J_\nu(bx)\,dx$$
$$=2^{-p}b^{-\frac{3}{2}}a^{p+\frac{1}{2}}e^{-\frac{b^2}{2a^2}}W_{-p-\frac{1}{2}\nu-\frac{3}{4},\,\frac{1}{2}(\nu+\frac{1}{2})}\left(\frac{b^2}{a^2}\right),$$

$$R(a)>0,\ R(b)>0,\ -1<R(\nu)<\tfrac{1}{2}\ R(p+\nu)\geqslant-\tfrac{3}{2}. \qquad (3.1)$$

PARTICULAR CASES.

(i) $p=-1$.

$$\int_0^\infty x\phi_\nu(ax)J_\nu(bx)\,dx=\frac{2a^{\nu-1}}{b^{\nu+1}}e^{-\frac{b^2}{a^2}},\quad -\tfrac{1}{2}\leqslant R(\nu)<\tfrac{1}{2}, \qquad (3.2)$$

on using the formula

$$W_{m+\frac{1}{2},\,\pm m}(z)=z^{m+\frac{1}{2}}e^{-\frac{1}{2}z}.$$

For $\nu=0$, (3.2) becomes

$$\int_0^\infty x\phi_0(ax)J_0(bx)\,dx=\frac{2}{ab}e^{-\frac{b^2}{a^2}}. \qquad (3.3)$$

For $\nu=-\frac{1}{2}$, (3.2) reduces to the familiar formula

$$\int_0^\infty e^{-\frac{1}{4}a^2x^2}\cos bx\,dx=\frac{\pi^{\frac{1}{2}}}{a}e^{-\frac{b^2}{a^2}}. \qquad (\alpha)$$

* G. N. Watson: *Theory of Bessel Functions* (1922) § 13.3 (4).

† Whittaker and Watson: *Modern Analysis*, (1927) § 16.12.

(ii) $p=0$.

$$\int_0^\infty \phi_{\nu+1}(ax)J_\nu(bx)dx=a^{\frac{1}{2}}b^{-\frac{3}{2}}e^{-\frac{b^2}{2a^2}}W_{-\frac{1}{2}\nu-\frac{3}{4},\,\frac{1}{2}(\nu+\frac{1}{2})}\left(\frac{b^2}{a^2}\right),$$
$$-1<R(\nu)<\tfrac{1}{2}. \tag{3.4}$$

For $\nu=0$, this formula assumes the simpler form

$$\int_0^\infty \phi_1(ax)J_0(bx)dx=\frac{2}{b}e^{-\frac{b^2}{2a^2}}D_{-2}\left(\frac{b}{a}\sqrt{2}\right), \tag{3.5}$$

on using the formula

$$W_{k,\,\pm\frac{1}{4}}(z)=2^{-k+\frac{1}{4}}z^{\frac{1}{4}}D_{2k-\frac{1}{2}}(\sqrt{2z}).$$

For $\nu=-\frac{1}{2}$, we have

$$\int_0^\infty \phi_{\frac{1}{2}}(ax)\frac{\cos bx}{\sqrt{x}}dx=\left(\frac{\pi a}{2}\right)^{\frac{1}{2}}\frac{1}{b}e^{-\frac{b^2}{2a^2}}W_{-\frac{1}{2},\,0}\left(\frac{b^2}{a^2}\right). \tag{3.6}$$

(iii) $p=-\frac{1}{2}\nu-\frac{3}{4}$,

$$\int_0^\infty x^{\frac{1}{2}\nu+\frac{3}{4}}\phi_{\frac{1}{2}(\nu+\frac{1}{2})}(ax)J_\nu(bx)dx$$
$$=\frac{2^{\frac{1}{2}\nu+\frac{3}{4}}}{(b\pi a)^{\frac{1}{2}\nu+\frac{5}{4}}}e^{-\frac{b^2}{2a^2}}K_{\frac{1}{2}\nu+\frac{1}{4}}\left(\frac{b^2}{2a^2}\right),\quad -1<R(\nu)<\tfrac{1}{2}, \tag{3.7}$$

on using the formula

$$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}W_{0,\,\nu}(2z).$$

For $\nu=-\frac{1}{2}$, 0, (3.7) becomes

$$a\int_0^\infty \phi_0(ax)\cos bx\,dx=e^{-2a^2}K_0\left(\frac{b^2}{2a^2}\right), \tag{3.8}$$

$$\int_0^\infty x^{\frac{3}{4}}\phi_{\frac{1}{4}}(ax)J_0(bx)dx=2^{\frac{3}{4}}b^{-\frac{1}{2}}a^{-\frac{5}{4}}\pi^{-\frac{1}{2}}e^{-\frac{b^2}{2a^2}}K_{\frac{1}{2}}\left(\frac{b^2}{2a^2}\right). \tag{3.9}$$

(iv) $\nu=-\frac{1}{2}$

$$\int_0^\infty x^{-p-\frac{1}{2}}\phi_{\frac{1}{2}+p}(ax)\cos bx\,dx$$
$$=\left(\frac{a}{2}\right)^{p+\frac{1}{2}}\frac{\pi^{\frac{1}{2}}}{b}e^{-\frac{b^2}{2a^2}}W_{-p-\frac{1}{2},\,0}\left(\frac{b^2}{a^2}\right),\quad R(p)\geqslant-1. \tag{3.10}$$

Putting $p=-1$, $-\frac{1}{2}$ in this formula, we arrive at (α) and (3.8) again.

(v) $\nu = 0$.

$$\int_0^\infty x^{-p}\phi_{p+1}(ax)J_0(bx)\,dx$$

$$= \frac{2a^p}{b}e^{-\frac{b^2}{2a^2}}D_{-2p-2}\left(\frac{b}{a}\sqrt{2}\right), \quad R(p) \geqslant -\tfrac{3}{2}. \qquad (3.11)$$

For $p = -\frac{3}{2}$ this formula reduces to the familiar result

$$\int_0^\infty xe^{-\frac{1}{4}a^2x^2}J_0(bx)\,dx = \frac{2}{a^2}e^{-\frac{b^2}{a^2}}.$$

Putting $p = -1, 0$ in (3.11) we arrive at (3.3) and (3.5).

4. In the same way we can prove the formula

$$\int_0^\infty x\phi_m(ax)\phi_m(bx)\,dx = \frac{2}{(ab)^{m+1}}\Gamma(-m)(a^2+b^2)^m,$$

$$R(a) > 0,\ R(b) > 0,\ -\tfrac{1}{2} < R(m) < 0. \qquad (4.1)$$

In particular, we have

$$\int_0^\infty x\phi_m^2(ax)\,dx = \frac{2^{m+1}}{a^2}\Gamma(-m). \qquad (4.2)$$

It is equally easy to prove the formulae

$$\int_0^\infty x^{\frac{1}{2}}\phi_{\frac{1}{2}}(ax)\,{\sin \atop \cos}\,\tfrac{1}{4}bx^2\,dx$$

$$= \frac{1}{\sqrt{ab}}\left[\pm\tfrac{1}{2}\log\frac{a^2+a\sqrt{2b}+b}{a^2-a\sqrt{2b}+b} + \tan^{-1}\frac{a\sqrt{2b}}{b-a^2}\right],$$

$$R(a) > 0,\ R(b) > 0. \qquad (4.3)$$

Particular cases are

$$\int_0^\infty x^{\frac{1}{2}}\phi_{\frac{1}{2}}(ax)\,{\sin \atop \cos}\,\tfrac{1}{4}a^2x^2\,dx = a^{-\frac{3}{2}}\left[\frac{\pi}{2} + \log(\sqrt{2}\pm 1)\right],$$

$$R(a) > 0. \qquad (4.4)$$

ON HADAMARD'S FACTORIZATION THEOREM*

BY

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1. In a paper published in 1927, Landau† gave a new proof of Hadamard's theorem on the factorization of an integral function of finite order. His proof arose out of the study of the problem of factorization of $(s-1)\zeta(s)$, which had been for the first time solved by Hadamard and for which Ananda Rau‡ gave an alternative proof by the use of Cauchy's theorem of residues. The object of this paper is to combine the methods of Landau and Ananda Rau to obtain another proof of Hadamard's general theorem of factorization. It is hoped that the method of this paper will have some interest, as it is in keeping with Cauchy's original proof§ of the infinite product for sin z as described in well-known text-books.

2. Let $f(z)$ be an integral function of finite order ρ. Let the zeros of the function, arranged in non-decreasing order of moduli, and each counted according to its degree of multiplicity, be $z_1, z_2, \ldots, z_n, \ldots$

Let $f(0) \neq 0$; let N be the integer defined by the property

$$|z_N| \leqslant R, \ |z_{N+1}| > R,$$

* I am deeply indebted to Prof. K. Ananda Rau for his kind guidance in the preparation of this paper.

† E. LANDAU: *Math. Zeit.* 26 (1927), 170-5.

‡ K. ANANDA RAU: The Infinite Product for $(s-1)\zeta(s)$, *Math. Zeit.* 20 (1924), 156-64.

§ E. GOURSAT: *Math. Analysis*, Vol. II, Pt, I, p. 143.

that N is a function of R tending to infinity with R. Then Hadamard's theorem states that

$$f(z) = e^{Q(z)}G(z),$$

where $G(z)$ is the canonical product with the zeros $z_1, z_2, z_3, \ldots$ and $Q(z)$ is a polynomial of degree not greater than ρ.

For the proof of the theorem we require the following

LEMMA. *Let* $0 < \varepsilon < 1$, $|z| \leqq R/2$;

then
$$\left| \frac{f'(z)}{f(z)} - \sum_{n=1}^{N} \frac{1}{z - z_n} \right| < cR^{\rho-1+\varepsilon},$$

where c is a positive numerical constant.

PROOF. Given a fixed ε, and $0 < \varepsilon < 1$, there exists an R_0 such that for $R > R_0$, we have

(i) $|f(z)| < e^{R^{\rho+\varepsilon}}$

(ii) $\log \dfrac{1}{|f(0)|} < (2R)^{\rho+\varepsilon}$

(iii) no z_n lies on $|z| = R/2$,

and (iv) $R > 4|x|$.

Now, for any $R > R_0$, let

$$g_R(z) = \frac{f(z)}{f(0)} \prod_{n=1}^{N} \left(1 - \frac{z}{z_n}\right)^{-1};$$

then, for $|z| = 2R$, we have,

$$|f(z)| < e^{(2R)^{\rho+\varepsilon}}, \quad \left|1 - \frac{z}{z_n}\right| \geqslant 1, \text{ where } 1 \leqslant n \leqslant N.$$

Therefore,
$$|g_R(z)| < \frac{1}{|f(0)|} e^{(2R)^{\rho+\varepsilon}}.$$

By the maximum-modulus theorem, this holds for $|z| = R$. Hence, for $|z| = R$,

$$\log |g_R(z)| < \log \frac{1}{|f(0)|} + (2R)^{\rho+\varepsilon} < 2^{\rho+2}R^{\rho+\varepsilon}.$$

Let $h_R(z) = \operatorname{Log} g_R(z)$ be that branch of the logarithm which vanishes for $z = 0$. $g_R(z)$ is analytic, and does not

vanish for $|z| \leqslant R$. Therefore, $h_R(z)$ is analytic for $|z| \leqslant R$, and $h_R(0)=0$. For $|z|=R$,

$$\text{real part of } h_R(z) = \text{arith. log } |g_R(z)| < 2^{\rho+2}R^{\rho+\varepsilon}.$$

By Carathéodory's inequality, for $|z|=R/2$, we have

$$|h'_R(z)| < cR^{\rho-1+\varepsilon}.$$

That is
$$\left|\frac{f'(z)}{f(z)} - \sum_{1}^{N} \frac{1}{z-z_n}\right| < cR^{\rho-1+\varepsilon}.$$

3. Let μ denote the greatest integer contained in ρ, so that $\mu \leqslant \rho$. Consider the integral,

$$I = \int \frac{f'(z)}{f(z)} \frac{1}{z^{\mu}(z-x)} dz,$$

taken along the circle $|z|=R/2$. The only poles of the integrand are the zeros of $f(z)$ lying inside the circle $|z|=R/2$, as also 0 and x.

Calculating the residues, we get,

$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} \frac{dz}{z^{\mu}(z-x)} = \frac{f'(x)}{f(x)} \frac{1}{x^{\mu}} + \sum_{|z_n|<R/2} \frac{1}{z_n^{\mu}(z_n-x)}$$
$$+ \frac{1}{(\mu-1)!}\left(\frac{d^{\mu-1}}{dz^{\mu-1}}\right)_{z=0}\left(\frac{f'(z)}{f(z)} \frac{1}{z-x}\right). \qquad (3.1)$$

We shall now prove that the left side of (3.1) tends to zero when $R \to \infty$ through such a sequence of values as will make $|z|=R/2$ free from any z_n. For,

$$I = \int\left(\frac{f'(z)}{f(z)} - \sum_{n=1}^{N} \frac{1}{z-z_n}\right)\frac{dz}{z^{\mu}(z-x)} + \sum_{1}^{N}\int \frac{1}{z-z_n} \frac{dz}{z^{\mu}(z-x)} \qquad (3.2)$$
$$= I_1 + I_2,$$

the path of integration being understood as $|z|=R/2$.

From the lemma proved above, we obtain, for $|z|=R/2$,

$$|I_1| = O(R^{\rho-\mu-1+\varepsilon}) = o(1), \qquad (3.3)$$

by our hypothesis on μ.

Let k denote the number of z_n's lying inside $|z|=R/2$, then

$$I_2=\sum_{n=1}^{k}\int_{|z|=R/2}\frac{1}{z-z_n}\frac{dz}{z^\mu(z-x)}+\sum_{n=k+1}^{N}\int_{|z|=R/2}\frac{1}{z-z_n}\frac{dz}{z^\mu(z-x)}$$

$$=J_1+J_2, \text{ say.} \tag{3.4}$$

The value of J_1 remains unaltered when integrated along $|z|=3R/4$, since the ring-shaped region enclosed between the two circles $|z|=R/2$ and $|z|=3R/4$ does not contain any poles of the integrand. Along the new path,

$$|z-z_n|\geqslant R/4,\ n\leqslant k,$$

and since

$$N=O(R^{\rho+\varepsilon}),$$

we have

$$|J_1|=O(R^{\rho-\mu-1+\varepsilon})=o(1) \tag{3.5}$$

Similarly, the value of J_2 remains unaltered when integrated along $|z|=R/4$, and we have,

$$|J_2|=O(R^{\rho-\mu-1+\varepsilon})=o(1). \tag{3.6}$$

Hence we deduce that

$$|I|=o(1). \tag{3.7}$$

We have now proved that,

$$\frac{f'(x)}{f(x)}\frac{1}{x^\mu}+\frac{1}{(\mu-1)!}\left(\frac{d^{\mu-1}}{dz^{\mu-1}}\right)_{z=0}\left[\frac{f'(z)}{f(z)}\frac{1}{z-x}\right]+\sum_{n=1}^{\infty}\frac{1}{z_n^\mu(z_n-x)}=0. \tag{3.8}$$

Let $\phi(z)=f'(z)/f(z)$. Then, by Leibnitz's theorem,

$$\frac{d^{\mu-1}}{dz^{\mu-1}}\left(\frac{\phi(z)}{z-x}\right)=\frac{1}{z-x}\phi^{(\mu-1)}(z)-{}_{\mu-1}C_1\frac{\phi^{(\mu-2)}(z)}{(z-x)^2}+\cdots$$
$$+(-1)^{\mu-1}\frac{(\mu-1)!}{(z-x)^\mu}\phi(z).$$

Therefore

$$\frac{1}{(\mu-1)!}\left(\frac{d^{\mu-1}}{dz^{\mu-1}}\right)_{z=0}\left(\frac{\phi(z)}{z-x}\right)=\sum_{1}^{\mu}a_nx^{-n},$$

where

$$a_n=-\phi^{(\mu-n)}(0)/(\mu-n)!$$

Hence, from (3·8), we obtain

$$\frac{f'(x)}{f(x)}\frac{1}{x^{\mu}}+\sum_{n=1}^{\infty}\frac{1}{z_n^{\mu}(z_n-x)}+\sum_{1}^{\mu}a_n x^{-n}=0,$$

or, $$\frac{f'(x)}{f(x)}+\sum_{n=1}^{\infty}\left(\frac{x}{z_n}\right)^{\mu}\frac{1}{z_n-x}+\sum_{0}^{\mu-1}b_n x^{n}=0,\text{ where } b_n=a_{\mu-n}.$$

That is,

$$\frac{f'(x)}{f(x)}=\sum_{n=1}^{\infty}\left(\frac{x}{z_n}\right)^{\mu}\frac{1}{x-z_n}+\sum_{0}^{\mu-1}c_n x^{n},\text{ where } c_n=-b_n. \tag{3·9}$$

Now $$\frac{1}{x-z_n}=\frac{1}{x}+\frac{z_n}{x^2}+\dots+\frac{z_n^{\mu-1}}{x^{\mu}}+\frac{1}{(x-z_n)}\left(\frac{z_n}{x}\right)^{\mu}.$$

Therefore

$$\left(\frac{x}{z_n}\right)^{\mu}\frac{1}{x-z_n}=\frac{x^{\mu-1}}{z_n^{\mu}}+\frac{x^{\mu-2}}{z_n^{\mu-1}}+\dots+\frac{1}{z_n}+\frac{1}{x-z_n}.$$

Substituting this in (3·9), and integrating, and raising each side to the power of e we get the required result, viz.

$$f(x)=e^{\sum_{0}^{\mu}d_n x^n}\prod_{n=1}^{\infty}\left(1-\frac{x}{z_n}\right)e^{\frac{x}{z_n}+\frac{x^2}{2z_n^2}+\dots+\frac{x^{\mu}}{\mu z_n^{\mu}}};$$

where $$d_n=c_{n-1}/n=\phi^{(n-1)}(0)/n!.$$

If $f(0)=0$, we can write $f(z)=z^{\alpha}\psi(z)$, and apply a similar method for the function $\psi(z)$.

A BI-VARIATE CORRELATED GAMMA-TYPE DISTRIBUTION FUNCTION

BY

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1. In this paper an expression for a correlated Gamma-type distribution function in two variables, with a definite value of the correlation coefficient is derived. Another correlated Gamma-type distribution function in which the regression is linear under all conditions has been given as a series bilinear in Laguerre polynomials by W. F. Kibble*.

The method employed is as follows. It is well known that if $x_i (i = 1, 2, \ldots, n)$ be n variables which vary normally and independently about zero with unit variance, then $\sum_{i=1}^{n} x_i^2 = \chi^2$ is distributed according to the law

$$df = \text{const. } (\chi^2)^{(n-2)/2} e^{-\chi^2/2} \, d(\chi^2/2).$$

If we put $\chi^2/2 = u$, we have, $df = \text{const. } u^{(n-2)/2} e^{-u} du$. Hence, if we have a number of normally distributed variates with zero mean and unit variance, by squaring and adding n of them we get a new variate with distribution defined by the function, $y = \text{const. } x^p e^{-x}$, where $p = (n-2)/2$.

Let u_1, u_2, u_3 be variables which are the sums of n_1, n_2, n_1 different variables of type x^2. Then u_1, u_2, u_3 are uncorrelated and their trivariate probability distribution function can be written as

$$K u_1^m e^{-u_1} du_1 . u_2^p e^{-u_2} du_2 . u_3^m e^{-u_3} du_3,$$

where $m = (n_1 - 2)/2$ and $p = (n_2 - 2)/2$.

* *Sankhya*, 5 (1941), 141-54.

Changing the variables by us'ng the substitution,

$$x = u_1 + u_2,\ y = u_2 + u_3,$$

and eliminating u_2 by integrating throughout its range of variation, we get

$$f_1(x, y)\,dxdy$$
$$= Ke^{-(x+y)}dxdy\int_0^x (x-u_2)^m\,(y-u_2)^m u_2^p e^{u_2} du_2,\ (x > y) \qquad (1)$$

and, $f_2(x, y)\,dxdy$

$$= Ke^{-(x+y)}dxdy\int_0^y (x-u_2)^m\,(y-u_2)^m u_2^p e^{u_2} du_2,\ (x > y). \qquad (2)$$

The function $\int_0^x u^{s-1}e^u du$ has an expansion similar to that of the incomplete Gamma function. For,

$$\int_0^x u^{s-1}e^u du = x^s e^x\left[\frac{1}{s} - \frac{x}{s(s+1)} + \cdots + \frac{(-1)^k x^k}{s(s+1)\ldots(s+k)} + (-1)^{k+1}\frac{\int_0^x u^{s+k}e^u du}{s(s+1)\ldots(s+k)}\right].$$

The remainder after $(k+1)$ terms tends to zero as k tends to infinity, the infinite series is absolutely convergent for all real values of x, and we have

$$(-1)^s\gamma(s, -x) = \int_0^x u^{s-1}e^u du = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s)}{\Gamma(s+k+1)} e^x x^{s+k}. \qquad (3)$$

Using (3) in (1) and (2), we get,

$$f_1(x,y) = K\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{m}{i}\binom{m}{j}\frac{\Gamma(i+j+p+1)}{\Gamma(i+j+p+k+2)} \times e^{-y}y^{m-j}x^{m+j+p+k+1}. \qquad (4)$$

$$f_2(x,y) = K\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{m}{i}\binom{m}{j}\frac{\Gamma(i+j+p+1)}{\Gamma(i+j+p+k+2)} \times e^{-x}x^{m-i}y^{m+i+p+k+1}. \qquad (5)$$

Equations (4) and (5) give the bivariate distribution of x and y. The marginal distributions are of the form,

$$dp = \lambda z^M e^{-z} dz,$$

λ being a constant, $M = (n_1 + n_2 - 2)/2 = m + p + 1$.

If ρ be the correlation between x and y, then,

$$\rho = \frac{n_2}{n_1 + n_2} = \frac{p+1}{m+p+2}. \qquad (6)$$

2. *Moment Generating Functions.* The moment generating function $G(\alpha, \beta)$ of the distribution function $F(x, y)$ is defined as the integral

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(x, y) e^{\alpha x + \beta y} dx dy$$

and the general product moment about the origin μ_{rs} is the coefficient of $\alpha^r \beta^s / r!\, s!$ in $G(\alpha, \beta)$. In what follows, it is assumed that $0 < \alpha < 1$, $0 < \alpha + \beta < 1$. This obviously does not involve any further restrictions. Here we have,

$F(x, y) = f_1(x, y)$ if $x < y$, and $F(x, y) = f_2(x, y)$ if $x > y$. Also the range of variation of both the variables is from zero to infinity. Hence,

$$G(\alpha, \beta) = \int_0^\infty dx \int_0^x f_2(x, y) e^{\alpha x + \beta y} dy + \int_0^\infty dy \int_0^y f_1(x, y) e^{\alpha x + \beta y} dx.$$

Substituting for $f_1(x, y)$, $f_2(x, y)$, from (4) and (5), we get the typical term of the m. g. fn. apart from a constant factor, as

$$\int_0^\infty e^{-(1-\alpha)x} x^{m-i} dx \int_0^x e^{\beta y} y^{m+i+p+k+1} dy + \int_0^\infty e^{-(1-\beta)y} y^{m-j} dy \times \int_0^y e^{\alpha x} x^{m+j+p+k+1} dx.$$

Using the expansion in (3) for the integrals involved we have

$$G(\alpha, \beta) = K \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{i+j+k+l} \binom{m}{i} \binom{m}{j} \times \frac{\Gamma(i+j+p+1)}{\Gamma(i+j+p+k+2)} \frac{\Gamma(2m+p+k+l+3)}{(1-\alpha-\beta)^{2m+p+k+l+3}}$$
$$\times \left\{ \beta^l \frac{\Gamma(m+i+p+k+2)}{\Gamma(m+i+p+k+l+3)} + \alpha^l \frac{\Gamma(m+j+p+k+2)}{\Gamma(m+j+p+k+l+3)} \right\}. \qquad (7)$$

The constant K is obtained either by integrating $f_1(x, y), f_2(x, y)$ throughout the x, y plane and equating to unity, or by putting $G(0, 0) = 1$.

Particular Cases.

(*a*) Let $m = 0$, and therefore $n_1 = 2$, i.e. both u_1, u_3 follow the exponential distribution law, and u_2 the Type III law. The distribution function of x and y is,

$$f_1(x, y) = Ke^{-(x+y)}dxdy\int_0^x u_2^p e^{u_2}du_2 \quad (x < y) \tag{8}$$

$$f_2(x, y) = Ke^{-(x+y)}dxdy\int_0^y u_2^p e^{u_2}du_2 \quad (x > y). \tag{9}$$

Again using (3), we have

$$f_1(x, y) = K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(p+1)}{\Gamma(p+k+2)}e^{-y}x^{p+k+1} \tag{10}$$

$$f_2(x, y) = K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(p+1)}{\Gamma(p+k+2)}e^{-x}y^{p+k+1}. \tag{11}$$

The marginal distribution is of the form $dp = \lambda z^{n_2/2}e^{-z}dz$, and ρ the correlation between x and y is $(p+1)/(p+2)$. The typical term of the m. g. fn. is

$$\int_0^\infty dx\int_0^x e^{-x}y^{p+k+1}e^{\alpha x+\beta y}dy + \int_0^\infty dy\int_0^y e^{-y}x^{p+k+1}e^{\alpha x+\beta y}dx$$

$$= \sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(p+k+2)}{(1-\alpha-\beta)^{p+k+l+3}}(\alpha^l+\beta^l).$$

Hence the m. g. fn. of the distribution function given by (10) and (11) is

$$G(\alpha, \beta) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\alpha^l+\beta^l}{(1-\alpha-\beta)^{p+k+l+3}} \tag{12}$$

$$= \frac{1}{(1-\alpha-\beta)^{p+1}(1-\alpha)(1-\beta)}. \tag{13}$$

The constant is 1, since (10), (11) and (13) have been derived directly by integration, but it is easily seen that they can be obtained by putting $m = i = j = 0$ in (4), (5) and (7) respectively.

(*b*) When $m=0$ and $p=0$, then $n_1=n_2=2$. And u_1, u_2, u_3 are all distributed in the exponential curve. In this case the marginal distributions are of the form $dp=\lambda z e^{-z} dz$. The joint distribution of x and y is,

$$ke^{-y}(1-e^{-x})\,dx\,dy, \quad x<y \tag{14}$$

$$ke^{-x}(1-e^{-y})\,dx\,dy, \quad x>y. \tag{15}$$

The correlation between x and y is $\frac{1}{2}$.

The m. g. fn. of the distribution is given by

$$G(\alpha, \beta)=\frac{1}{(1-\alpha-\beta)(1-\alpha)(1-\beta)}\,. \tag{16}$$

As before, it is clear that by putting $p=0$ in (10), (11) and (13), we get the above equations.

If μ_{rs} is the coefficient of $\alpha^r\beta^s/r!\,s!$ in the expansion of $G(\alpha, \beta)$, then,

$$\mu_{rs}=r!\,s!\Big[\binom{r+s}{s}+\binom{r+s-1}{s}+\binom{r+s-1}{s-1}+\binom{r+s-2}{s}+\binom{r+s-2}{s-1}+\binom{r+s-2}{s-2}+\ldots\Big]. \tag{17}$$

Therefore,

$$\frac{\mu_{rs}}{r!\,s!}-\frac{\mu_{r-1,\,s-1}}{(r-1)!\,(s-1)!}=\Big[\binom{r+s}{s}+\binom{r+s-1}{s}+\binom{r+s-1}{r}+\ldots+\binom{s}{s}+\binom{r}{r}\Big]. \tag{18}$$

And

$$\frac{\mu_{rs}}{r!\,s!}-\frac{\mu_{r-1,\,s}}{(r-1)!\,s!}-\frac{\mu_{r,\,s-1}}{r!\,(s-1)!}+\frac{\mu_{r-1,\,s-1}}{(r-1)!\,(s-1)!}=\frac{(r+s)!}{r!\,s!}. \tag{19}$$

3. *Regression Curves.* A linear regression between correlated variables is rather exceptional. The moments of each array in the case of this distribution function are easily obtained from the m. g. fn. of the distribution curves in each array. If $f_1(x, y)$ and $f_2(x, y)$ are the two functions defined in the two halves of the (x, y)-plane, then the m. g. fn. of the y-array for a fixed x is

$$\frac{\int_0^x e^{\alpha y} f_2(x, y)\,dy+\int_x^\infty e^{\alpha y} f_1(x, y)\,dy}{\int_0^x f_2(x, y)\,dy+\int_x^\infty f_1(x, y)\,dy} \tag{20}$$

Substituting from (4) and (5), for $f_1(x, y)$, $f_2(x, y)$ we get the typical term of the numerator of the m. g. fn. as

$$e^{-x}x^{m-i}\int_0^x e^{ay}y^{m+i+p+k+1}dy+\int_x^\infty e^{-(1-a)y}y^{m-j}dy.x^{m+j+p+k+1} \quad (21)$$

and the denominator as

$$e^{-x}x^{m-i}\int_0^x y^{m+i+p+k+1}dy+\int_x^\infty e^{-y}y^{m-j}dy\ x^{m+j+p+k+1} \quad (22)$$

Since

$$\int_x^\infty u^{s-1}e^{-u}du=\Gamma(s)-\gamma(s, x)=\Gamma(s)-\sum_{l=0}^\infty \frac{\Gamma(s)}{\Gamma(s+l+1)}e^{-x}x^{s+l},$$

using (3) and simplifying, we have,

$$e^{-x}\sum_{l=0}^\infty(-1)^l\frac{\Gamma(m+i+p+k+2)}{\Gamma(m+i+p+k+l+3)}e^{ax}a^lx^{2m+p+k+l+2}$$

$$+\left[\frac{\Gamma(m-j+1)}{(1-a)^{m-j+1}}x^{m+j+p+k+1}\right.$$

$$\left.-\sum_{l=0}^\infty\frac{\Gamma(m-j+1)}{\Gamma(m-j+l+2)}e^{-(1-a)x}(1-a)^lx^{2m+p+k+l+2}\right]$$

and

$$e^{-x}\frac{x^{2m+p+k+2}}{m+i+p+k+2}+x^{m+j+p+k+1}\Gamma(m-j+1)$$

$$-\sum_{l=0}^\infty\frac{\Gamma(m-j+1)}{\Gamma(m-j+l+2)}e^{-x}x^{2m+p+k+l+2}.$$

Hence the numerator of the m. g. fn. can be written as

$$\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty(-1)^{i+j+k}\binom{m}{i}\binom{m}{j}\frac{\Gamma(i+j+p+1)}{\Gamma(i+j+p+k+2)}$$

$$\left[e^{-(1-a)x}x^{2m+p+k+2}\sum_{l=0}^\infty\left\{(-1)^l\frac{\Gamma(m+i+p+k+2)}{\Gamma(m+i+p+k+l+3)}a^lx^l\right.\right.$$

$$\left.\left.-\frac{\Gamma(m+j+1)}{\Gamma(m+j+l+2)}(1-a)^lx^l\right\}+\frac{\Gamma(m-j+1)}{(1-a)^{m-j+1}}x^{m+j+p+k+1}\right]. \quad (23)$$

And the denominator becomes

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{m}{i}\binom{m}{j}\frac{\Gamma(i+j+p+1)}{\Gamma(i+j+p+k+1)}$$

$$\left[e^{-x}x^{2m+p+k+1}\left\{\frac{1}{m+i+p+k+2}-\sum_{l=0}^{\infty}\frac{\Gamma(m-j+1)}{\Gamma(m-j+l+2)}x^{l}\right\}\right.$$

$$\left.+\Gamma(m-j+1)\,x^{m+j+p+k+1}\right]. \quad (24)$$

It is clear that (23) reduces to (24) by putting $a=0$.

(*a*) When $m=0$, the m. g. fn. of an y-array for a fixed x is given by

$$\frac{\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(p+1)}{\Gamma(p+k+2)}\left[e^{-x}\int_{0}^{x}e^{ay}y^{p+k+1}dy+x^{p+k+1}\int_{x}^{\infty}e^{-(1-a)y}dy\right]}{\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(p+1)}{\Gamma(p+k+2)}\left[e^{-x}\int_{0}^{x}y^{p+k+1}dy+x^{p+k+1}\int_{x}^{\infty}e^{-y}dy\right]} \quad (25)$$

which by a similar integration reduces to

$$e^{ax}\frac{\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(p+1)}{\Gamma(p+k+2)}}{\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(p+1)}{\Gamma(p+k+2)}}$$

$$\times\frac{\sum_{l=0}^{\infty}(-1)^{l}\frac{\Gamma(p+k+2)}{\Gamma(p+k+l+3)}a^{l}x^{p+k+l+2}+\frac{x^{p+k+1}}{(1-a)}}{\frac{x^{p+k+2}}{p+k+2}+x^{p+k+1}}. \quad (26)$$

It can be shown that the ratio of (23) to (24) reduces to (26) by putting, $m=i=j=0$. By making this substitution, (23) becomes

$$\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(p+1)}{\Gamma(p+k+2)}\left[e^{-(1-a)x}x^{p+k+2}\right.$$

$$\left.\times\sum_{l=0}^{\infty}x^{l}\left\{(-1)^{l}\frac{\Gamma(p+k+2)}{\Gamma(p+k+l+3)}a^{l}-\frac{(1-a)^{l}}{\Gamma(l+2)}\right\}+\frac{x^{p+k+1}}{(1-a)}\right].$$

The second term in this expression is

$$-\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(p+1)}{\Gamma(p+k+2)}e^{-(1-a)x}x^{p+k+2}\sum_{l=0}^{\infty}\frac{(1-a)^l x^l}{\Gamma(l+2)}$$

$$=-\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(p+1)}{\Gamma(p+k+2)}\left[\frac{x^{p+k+1}}{1-a}-\frac{x^{p+k+1}e^{-(1-a)x}}{(1-a)}\right].$$

Hence (22)

$$=e^{-(1-a)x}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(p+1)}{\Gamma(p+k+2)}$$

$$\times\left\{\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(p+k+2)}{\Gamma(p+k+l+3)}a^l x^{(p+k+l+2)}+\frac{x^{p+k+1}}{(1-a)}\right\},$$

which is the numerator in (26). A similar transformation of (24) gives the denominator in (26).

(*b*) When $m=p=0$, the m. g. fn. is given by

$$\frac{e^{-x}\int_0^x e^{ay}(1-e^{-y})dy+(1-e^{-x})\int_x^\infty e^{-(1-a)y}dy}{e^{-x}\int_0^x(1-e^{-y})dy+(1-e^{-x})\int_x^\infty e^{-y}dy}, \tag{27}$$

which reduces to

$$\frac{e^{ax}-1}{ax(1-a)}. \tag{28}$$

This result is obtained by putting $p=0$ in (26). We get

$$e^{ax}\frac{\sum_{k=0}^{\infty}(-1)^k\frac{1}{\Gamma(k+2)}\left\{\sum_{l=0}^{\infty}(-1)^l\frac{\Gamma(k+2)}{\Gamma(k+l+3)}a^l x^{k+l+2}+\frac{x^{k+1}}{1-a}\right\}}{\sum_{k=0}^{\infty}(-1)^k\frac{1}{\Gamma(k+2)}\left\{\frac{x^{k+2}}{k+2}+x^{k+1}\right\}}$$

The first term in the numerator of this expression is

$$\sum_{k=0}^{\infty}(-1)^{k+2}\frac{x^{k+2}}{(k+2)!}+a\sum_{k=0}^{\infty}(-1)^{k+3}\frac{x^{k+3}}{(k+3)!}+\dots\text{ etc.},$$

which sums up to $\dfrac{ae^{-x}-e^{-ax}-a+1}{(1-a)a}$.

Similarly the other terms sum to $\dfrac{e^{\alpha x}-1}{\alpha x(1-\alpha)}$.

The rth moment about the origin of the distribution in each array is

$$\mu'_r = r!\left[1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots+\frac{x^r}{(r+1)!}\right], \qquad (29)$$

from which we have the recurrence relation,

$$(\mu'_r - r\mu'_{r+1}) = \frac{x_r}{(r+1)}. \qquad (30)$$

4. The distributions which have been investigated above in a theoretical manner can be illustrated experimentally. The deviations of Tippet's random sampling numbers from their mean as given by Mahalanobis* are distributed normally with unit standard deviation. Therefore their squares are distributed in the χ^2 curve. These squares have been calculated by A. N Krishnan Nair†. They were chosen in groups of N, and x and y, where

$N = 2n_1+n_2$,

$x =$ sum of the first n_1+n_2 numbers,

$y =$ sum of the last n_2+n_1 numbers.

The frequencies were tabulated and the bivariate correlation table formed. Four such tables were prepared with the following values for the correlation coefficient ρ, n_1, n^2, m and p.

* P. C. Mahalanobis, Tables of Random Samples from a Normal population, *Sankhya*, 1 (1937), 289-328.

† "The Distribution of Student's Ratio and the correlation coefficient in samples from a certain non-normal population"—one of the papers included in a thesis submitted to the Madras University in March 1940.

Tables	ρ	n_1	n_2	m	p
I	$\frac{1}{2}$	2	2	0	0
II	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$
III	$\frac{2}{3}$	1	2	$-\frac{1}{2}$	0
IV	$\frac{3}{4}$	1	3	$-\frac{1}{2}$	$+\frac{1}{2}$

Table I corresponds to the particular case, 2 (*b*) where $m=p=0$, so that $n_1=n_2=2$. In this case the values of $\mu_{01}, \mu_{10}, \mu_{11}, \ldots$ were calculated for comparison with theoretical values. It is to be noted that the unit used in the derivation of the distribution is $u=\chi^2/2$, while the unit in the experimental work is χ^2. Hence for comparison, the values of μ_{rs} obtained from the tables should be divided by 2^{r+s}, for all r and s. The corrected values of the first few moments calculated from Table I are given below with the corresponding theoretical values obtained by expanding

$$G(\alpha, \beta) = (1-\alpha-\beta)^{-1}(1-\alpha)^{-1}(1-\beta)^{-1}$$
$$= 1+2(\alpha+\beta)+3(\alpha^2+\beta^2)+5\alpha\beta+\ldots$$

Values	μ_{01}	μ_{10}	μ_{11}	μ_{02}	μ_{20}
Theoretical	2	2	5	6	6
Experimental	2·0245	2.0005	5.0675	6.0550	6.0537.

It is seen that there is great similarity in the results.

In skew bivariate surfaces, the question of linearity of regression is one of great interest. In an exhaustive treatise on this subject, Pretorius* has discussed several cases of this type of frequency surface, and has shown that the means of the arrays are distributed more or less in a linear curve, if their values at the extremities of the distribution are omitted. As given in (29), regression is unmistakably linear in the particular case 2(*b*). In 2(*a*) also, the expression can be simplified if we assume that p

* S. J. Pretorius, On Skew bivariate surfaces, *Biometrika*, 22 (1930-31), 109-223.

takes only integral values. In this case the rth moment about the origin, of the y-arrays is given by

$$\mu'_r = r!\left[\frac{x^r}{r!}+\frac{x^{r-1}}{(r-1)!}\left(1-\frac{x}{p+2}\right)\right.$$
$$\left.+\left(1-\frac{x}{p+2}+\ldots+(-1)^r\frac{x^r}{(p+2)(p+3)\ldots(p+r+1)}\right)\right]$$
$$=r!\left[1+x\frac{p+1}{p+2}+\frac{x^2}{2!}\frac{p+1}{p+3}+\ldots+\frac{x^r}{r!}\frac{p+1}{p+r+1}\right]. \qquad (31)$$

Hence, $\mu'_1 = \left(1+\frac{p+1}{p+2}x\right)$,

$$\mu'_2 = 2!\left(1+\frac{p+1}{p+2}x+\frac{p+1}{p+3}\frac{x^2}{2!}\right), \text{ etc.} \qquad (32)$$

The curve of array-variances is given by

$$\sigma_y^2(x) = 1+x^2\frac{p+1}{(p+3)(p+2)^2}. \qquad (33)$$

If we put $p=0$ in (31), we get

$$\mu'_r = r!\left[1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots+\frac{x^r}{(r+1)!}\right],$$

which is equation (29).

Hence $$\mu'_1 = 1+\frac{x}{2}. \qquad (34)$$

and

$$\sigma_y^2(x) = 1+\frac{x^2}{12}. \qquad (35)$$

Thus, in these two cases, subject to the condition that p is an integer, we find that the curves of regression are linear while those of variances are of the second degree. The curve of values of $\sqrt{\sigma_y^2(x)}$ plotted to values of x is called the Scedastic curve. For purposes of comparison, curves were fitted by the method of orthogonal polynomials to the values of variances of arrays. The equations of the curves of good fit, for the four correlation tables are given below.

Table I	Regression	$y = 5.668 + 0.439\ \xi_1$
	Variances	$y = 11.388 + 1.705\ \xi_1 + 0.245\ \xi_2$
Table II	Regression	$y = 8.003 + 0.843\ \xi_1$
	Variances	$y = 3.852 + 0.228\ \xi_1 - 0.027\ \xi_2$
Table III	Regression	$y = 7.008 + 0.706\ \xi_1$
	Variances	$y = 5.972 + 1.109\ \xi_1 + 0.053\ \xi_2$
Table IV	Regression	$y = 6.494 + 0.478\ \xi_1$
	Variances	$y = 7.912 + 1.245\ \xi_1 + 0.119\ \xi_2$

$$\xi_1 = (x - \bar{x}) \text{ and } \xi_2 = \xi_1^2 - 4.$$

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THE DERIVATIVES OF AN ALGEBRAIC FUNCTION

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1. Let u be an integral algebraic function of z defined by an irreducible polynomial equation $f(z,u) = 0$ of degree n in u, and ρ in z. The coefficient of u^n in $f(z, u)$ is obviously independent of z, and

$$\frac{f(z, u)}{z^{n\rho}} = g\left(\frac{1}{z}, \frac{u}{z^\rho}\right),$$

where $g(\xi, \eta)$ is of degree n in η, and the coefficient of η^n is independent of ξ.

If the equations

$$f(a, u) = 0, \text{ for all finite } a \tag{1}$$

and

$$g(0, \eta) = 0 \tag{2}$$

have roots of multiplicity not greater than two, Beatty* has shown that

$$u'(z) = \sum_a \sum_b \frac{2\sigma f(a, u)}{(u-b)(z-a) f_{uu}(a, b)} + P(z), \tag{3}$$

where $P(z)$ is a polynomial of degree $\rho - 1$ at most, b runs through the finite multiple roots of (1), a through the associated values of z, and σ is the first power of $z-a$ to have different coefficients in the expansions of the several branches of $u-b$ in terms of $z-a$.

2. The object of this paper is to prove the above result by an independent method, and to extend it to the case when the multiplicity of the roots of equations

* The derivatives of an algebraic function, *Trans. Roy. Soc. Canada*, Section III (1931), pp. 79—82.

(1) and (2) is not greater than 3. We shall prove the following

THEOREM. *If the roots of* (1) *and* (2) *be of multiplicity not greater than three, we have*

$$u'(z)=\sum_{a}\sum_{b_2}\frac{1}{2}\left(\frac{u-b}{z-a}\right)$$

$$+\sum_{a}\sum_{b_{31}}\left[\frac{1}{3}\frac{u-b_{31}}{z-a}-\frac{1}{36}\frac{f_{uuuu}(a,b)}{f_{uuu}(a,b)}\frac{(u-b_{31})^2}{z-a}\right.$$

$$\left.+\frac{1}{9}\frac{f_u^{(1)}}{f^{(1)}}\frac{(u-b_{31})^2}{z-a}\right]+\sum_{a}\sum_{b_{32}}\frac{2}{3}\left(\frac{u-b_{32}}{z-a}\right)+P(z),$$

where b_2 *runs through all cycles of order* 2, b_{31} *through all cycles of order* 3 *beginning with* $(z-a)^{\frac{1}{3}}$, *and* b_{32} *through all cycles of order* 3 *beginning with* $(z-a)^{\frac{2}{3}}$.

PROOF. From the equation $f(z, u)=0$, we easily obtain by expanding $f_z(z, u)$ and $f_u(z, u)$ in powers of $z-a$,

$$u'(z)=\frac{f(a, u)+(z-a)^2\sum_{s=2}^{\infty}\left(\frac{1}{s!}-\frac{1}{(s-1)!}\right)(z-a)^{s-2}f^{(s)}(a, u)}{(z-a)\,f_u(a, u)+(z-a)^2\sum_{s=1}^{\infty}\frac{(z-a)^{s-1}}{s!}f_u^{(s)}(a, u)}. \tag{4}$$

Beatty's result is easily deduced from (4), for when $f_u(a, b)=0$, $f_{uu}(a, b)\neq 0$, we have at $u=b$

$$u'(z)=\frac{f(a, u)+(z-a)^2\,P_1(z-a, u)}{(z-a)(u-b)\,f_{uu}(a, b)\left[1+P_1(u, b)+\frac{z-a}{u-b}P_3(z-a, u)\right]},$$

where P_1, P_2, P_3 are polynomials in $z-a$ and u. The principal part of $u'(z)$ is given by

$$\sum_{a}\sum_{b}\frac{f(a, u)}{f_{uu}(a, b)\,(z-a)\,(u-b)}.$$

Also $u'(z)$ has a pole of order $\rho-1$ at most at infinity, since

$$\frac{du}{dz}=\rho z^{\rho-1}\eta-z^{\rho-2}\frac{d\eta}{d\xi}$$

and η is regular at $\xi=0$. Hence we obtain

$$u'(z)=\sum_a\sum_b\frac{f(a,u)}{f_{uu}(a,b)(u-b)(z-a)}+P(z),$$

where $P(z)$ is a polynomial of degree $\rho-1$ at most.

We now consider $u'(z)$ for cycles of order 3. Cycles of order 3 beginning with $(z-a)^{\frac{1}{3}}$ will arise when

$$f=0,\ f_u=0,\ f_{uu}=0,\ f_{uuu}\neq 0,\ f'\neq 0$$

at (a, b). Using (4), the principal part of $u'(z)$ at $u=b$ is

$$\frac{2f(a,u)}{f_{uuu}(u-b)^2(z-a)}\left[1-\frac{(u-b)f_{uuuu}}{3f_{uuu}}-\frac{2(z-a)}{(u-b)^2}\frac{f_u^{(1)}}{f_{uuu}}\right],$$

the derivatives being all considered at (a, b). This simplifies to

$$\frac{u-b}{3(z-a)}-\frac{1}{36}\frac{f_{uuuu}(a,b)}{f_{uuu}(a,b)}\cdot\frac{(u-b)^2}{z-a}+\frac{1}{9}\frac{f_u^{(1)}(u-b)^2}{f^{(1)}\,z-a}.$$

For cycles of order 3 beginning with $(z-a)^{\frac{2}{3}}$, we must have the additional conditions

$$f_z=0,\ f_u^{(1)}=0,\ f_{zz}\neq 0,\ \text{at } (a, b).$$

Using (4), the principal part of $u'(z)$ at $u=b$'s

$$\frac{2f(a,u)-(z-a)^2 f_{zz}(a,u)}{f_{uuu}(a,b)(u-b)^2(z-a)},$$

which is equal to

$$\frac{u-b}{3(z-a)}-\frac{z-a}{(u-b)^2}\cdot\frac{f_{zz}(a,b)}{f_{uuu}(a,b)},$$

which reduces again to the expression $2(u-b)/3(z-a)$ in virtue of the polynomial equation.

The theorem enunciated now follows. We note incidentally that the coefficient of u/z in $u'(z)$ is ρ.

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